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Degenerate Hopf Bifurcations in Discontinuous Planar Systems

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We study the stability of a singular point for planar discontinuous differential equations with a line of discontinuities. This is done, for the most generic cases, by computing some kind of Lyapunov constants. Our computations are based on the so called (R, θ, p, q)-generalized polar coordinates, introduced by Lyapunov, and they are essentially different from the ones used in the smooth case. These Lyapunov constants are also used to generate limit cycles for some concrete examples.

1. INTRODUCTION

In smooth planar differential equations the stability of a nondegenerate critical point with complex eigenvalues is reduced to the computation of the so called *Lyapunov constants*; see [3], for instance. Furthermore these

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constants give information about the *cyclicity* of the point, that is, on the maximum number of small periodic orbits which appear from this critical point via degenerate Hopf bifurcation.

The aim of this paper is to obtain some kind of Lyapunov constants for nondegenerate singular points of planar differential equations with a line of discontinuities L, which we will assume is $L = \{(x, y),\ y = 0\}$, for sake of simplicity. These type of differential equations appear frequently in applications; see, for instance, [3, 16].

Our first idea was to develop similar methods to the ones used in [2, pp. 449–254; 5, 11] for the smooth case. In order to point out the major difficulties that we have found in this extension to the discontinuous case and to explain how we have overcome them, we will give some preliminaries.

We consider equations of type

$$(\dot{x}, \dot{y}) = \begin{cases} (X^+(x,y),\ Y^+(x,y)) & \text{if } y \geq 0, \\ (X^-(x,y),\ Y^-(x,y)) & \text{if } y \leq 0, \end{cases} \tag{1}$$

where (X^+, Y^+) (resp. (X^-, Y^-)) is called the component equation of (1) in the upper (resp. lower) half plane, being $X^{\pm}$ and $Y^{\pm}$ real analytical functions in a neighbourhood of $(0, 0)$. We assume that both vector fields can be smoothly extended to an open neighbourhood of the half plane where they are defined.

The main property of the flow of a smooth system near a nondegenerate critical point, p, of focus type is that it turns around the point. For systems of type (1) the role of such points is taken by four types of singular points that, for short, we will call *pseudo-focus* and which we will describe in the sequel just assuming the case in which the flow turns around p counterclockwise. See also [10, Chap. 4].

(i) Points of *focus–focus* type at $p \in L$: both systems $(X^{\pm}, Y^{\pm})$ have a critical point at p with complex eigenvalues and their solutions turn around p counterclockwise.

(ii), (iii) Points of *focus–parabolic* (resp. *parabolic–focus*) type at $p \in L$: the system defined in the upper (resp. lower) half plane has a critical point of focus type at p while the solutions of the system defined in a neighbourhood of the lower (resp. upper) half plane have a parabolic contact (i.e., a second order contact point) with L at p, the solution of which at this contact is locally contained in the upper (resp. lower) plane. See Fig. 1.

(iv) Points of *parabolic–parabolic type* at $p \in L$: the solutions of both systems have a parabolic contact at p with L in such a way that the flow induced by (1) turns around p.

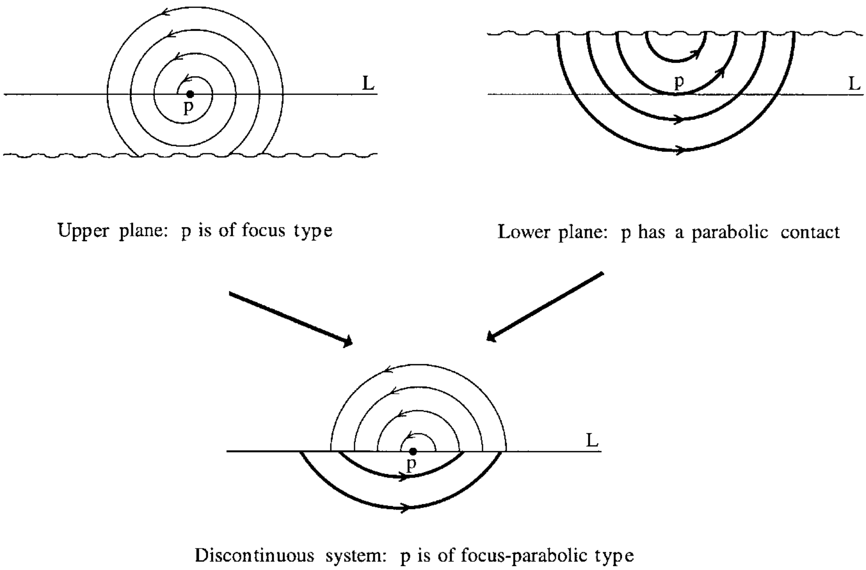

FIG. 1. A singular point of focus–parabolic type.

Observe that the parabolic–focus type can be reduced to the focus–parabolic case by applying the change of coordinates $(x, y, t) \to (-x, -y, t)$ to Eq. (1). Hence, for short, just the first case will be studied. We point out that the singularity of parabolic–parabolic type is also known as a *sewed focus* (see [10, p. 234]).

The main idea for determining the stability of a singular point p (which we will assume is $(0,0)$, again for the sake of simplicity) of Eq. (1) consists in starting from $(r_0, 0)$, $r_0 > 0$, r_0 small enough, and to evaluate the sign of the first nonzero term of $h^-(h^+(r_0)) - r_0$ in its r_0-power series expansion, where h^+ (resp. h^-) is the Poincaré return map associated with the flow of the corresponding component equation of (1) in $\{y \geq 0\}$ (resp. $\{y \leq 0\}$) defined between both sides of $L \setminus p$.

When, in the half plane that we study, the point p is of focus type, the ideas are the same as the ones used in the smooth case. The only difficulty appears from the length of the expressions involved. These expression are even larger than in the smooth case. In the smooth case some cancellations related to considering the flow giving a complete turn around the origin appear. In discontinuous planar systems, these cancellations do not appear because we consider just half turns. In any case the maps h^+ and h^- defined above are analytic.

The case in which the flow has a parabolic contact at p presents more difficulties. If we try to repeat the same ideas as in the focus case, that is, to write the equation in polar coordinates and to study the maps h^+ and h^-, it is not clear if these maps are analytic (see Fig. 2 and Section 5). In order to avoid this problem we use quasihomogeneous polar coordinates with suitable weights; see [6, 15]. By using these new coordinates we can assure that the corresponding return maps are also analytic and a similar treatment to the focus case can be done for this new case. See again Fig. 2. In any case the computations involved are not the standard ones and our approach is essentially different from that of [10, Chap. 4, Sect. 19.4].

At this point and because of the above considerations, it turns out that for a critical point of pseudo-focus type either $h^-(h^+(r_0)) - r_0 \equiv 0$ or that there exists a k such that $h^-(h^+(r_0)) - r_0 = V_k r_0^k + O(r_0^{k+1})$. In the first case the origin of (1) is a center; in the second case it is said that the origin is *a weak focus of order* k and that *its* kth *Lyapunov constant* is V_k. Observe that when an expression for V_k is given, it makes sense only when $V_1 = V_2 = \cdots = V_{k-1} = 0$. In the case of smooth differential equations, it is well known that this value k with $V_k \neq 0$ (if it exists) is an odd number; see [2, p. 243]. In the case of discontinuous differential equations k does not have to be necessarily odd as will be proved in next results.

The main goal of this paper is to obtain the general expressions for the first three Lyapunov constants for a critical point of pseudo-focus type for a differential equation of type (1).

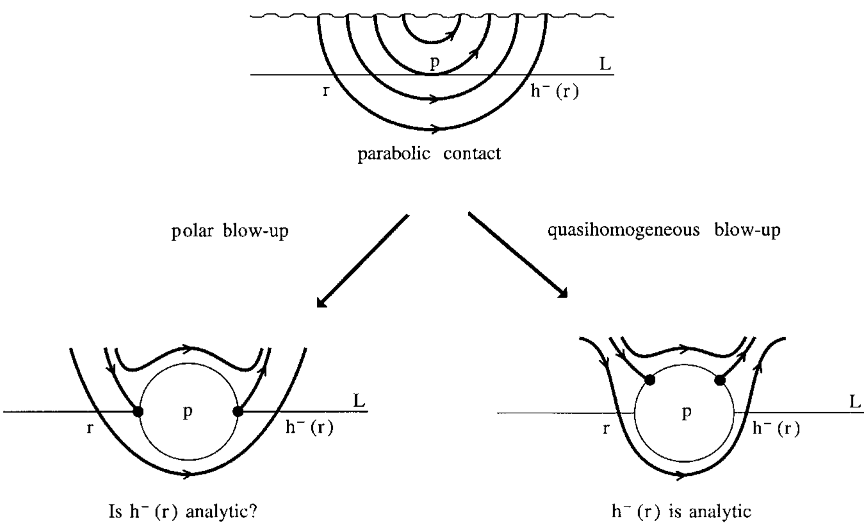

FIG. 2. Analyticity of the return map for parabolic contacts.

2. DEFINITIONS AND MAIN RESULTS

Taking complex coordinates $z = x + iy$, Eq. (1) is written as

$$\dot{z} = \begin{cases} F^{+}(z,\bar{z}) = a^{+} + \sum_{k=1}^{\infty} F_k^{+}(z,\bar{z}) & \text{if } \operatorname{Im}(z) \geq 0, \\ F^{-}(z,\bar{z}) = a^{-} + \sum_{k=1}^{\infty} F_k^{-}(z,\bar{z}) & \text{if } \operatorname{Im}(z) \leq 0, \end{cases} \tag{2}$$

where $F_k^{\pm}$ is a complex homogeneous polynomial of degree k in the variables z and $\bar{z}$,

$$F_k^{\pm}(z,\bar{z}) = \sum_{l+m=k} f_{lm}^{\pm} z^l \bar{z}^m, \tag{3}$$

and $a^{\pm} \in \mathbb{R}$. For system (1) we put

$$\begin{aligned} X^{\pm}(x,y) &= a^{\pm} + b^{\pm}x + c^{\pm}y + d^{\pm}x^2 + e^{\pm}xy + f^{\pm}y^2 + g^{\pm}x^3 + \cdots, \\ Y^{\pm}(x,y) &= l^{\pm}x + m^{\pm}y + n^{\pm}x^2 + o^{\pm}xy + p^{\pm}y^2 + q^{\pm}x^3 \\ &\quad + r^{\pm}x^2y + s^{\pm}xy^2 + t^{\pm}y^3 + u^{\pm}x^4 + \cdots, \end{aligned} \tag{4}$$

the dots being the remaining terms of $X^{\pm}$ or $Y^{\pm}$ in their power series expansion at $(0,0)$.

In accordance with previous notation and with the definition of critical points of pseudo-focus type, we will define the following types of component equations for Eq. (1).

Definition 1. The component equation (X^+, Y^+) (resp. (X^-, Y^-)) of Eq. (1) is of focus type (F) if $a^+ = 0$ (resp. $a^- = 0$), $(b^+ - m^+)^2 + 4l^+c^+ < 0$ (resp. $(b^- - m^-)^2 + 4l^-c^- < 0$), and $l^+ > 0$ (resp. $l^- > 0$). Observe that if complex coordinates are used then we have that the component equation of (1), (2) defined in $\operatorname{Im}(z) \geq 0$ (resp. $\operatorname{Im}(z) \leq 0$) is of type (F) if

$$a^{\pm} = 0, \qquad f_{01}^{\pm}\bar{f}_{01}^{\pm} - \left(\operatorname{Im}\left(f_{10}^{\pm}\right)\right)^2 < 0, \qquad \text{and} \qquad \operatorname{Im}\left(f_{10}^{\pm} + f_{01}^{\pm}\right) > 0. \tag{5}$$

Definition 2. The component equation (X^+, Y^+) (resp. (X^-, Y^-)) of Eq. (1) is of parabolic type (P) if $a^+ < 0$ (resp. $a^- > 0$) and $l^{\pm} > 0$. Note that, in complex coordinates, this means that the component equation of (1), (2) defined in $\operatorname{Im}(z) \geq 0$ (resp. $\operatorname{Im}(z) \leq 0$) is of type (P) if $a^+ < 0$ (resp. $a^- > 0$) and $\operatorname{Im}(f_{10}^{\pm} + f_{01}^{\pm}) > 0$.

In agreement with these definitions we have the following main results.

PROPOSITION A. *Let us consider Eq.* (2), *where* $F_k^{\pm}$ *is given by* (3). *Define*

$$\nu^{\pm} := \exp\left\{ \frac{\pi \operatorname{Re}(f_{10}^{\pm})}{\sqrt{\left(\operatorname{Im}(f_{10}^{\pm})\right)^2 - f_{01}^{\pm}\overline{f_{01}^{\pm}}}} \right\}. \tag{6}$$

Its first Lyapunov quantity at the origin, V_1, *is as follows. If* $(0,0)$ *is a singularity of*

(a) *focus–focus type, then* $V_1 = \nu^{+}\nu^{-} - 1$,
(b) *focus–parabolic type, then* $V_1 = \nu^{+} - 1$,
(c) *parabolic–parabolic type, then* $V_1 = 0$.

The relations between the coefficients of Eqs. (1) and (2), involved in former proposition, with $\nu^{\pm}$ are the following: $f_{10}^{\pm} = (b^{\pm} + m^{\pm} + (l^{\pm} - c^{\pm})i)/2$, $f_{01}^{\pm} = (b^{\pm} - m^{\pm} + (l^{\pm} + c^{\pm})i)/2$, and

$$\nu^{\pm} = \exp\left\{ \pi(b^{\pm} + m^{\pm}) \Big/ \sqrt{(-1)\left((b^{\pm} - m^{\pm})^2 + 4l^{\pm}c^{\pm}\right)} \right\}.$$

To go further in the calculations of the Lyapunov constants we make the following assumption. If in, Eq. (2), we have any component equation of type (F), then we suppose that $F_1(z, \bar{z}) = (i + \lambda)z$, with $\lambda \in \mathbb{R}$, for this component equation. In fact what we are assuming is that this focus is written in its real Jordan form. This is a restriction but, as we will see, in this case the computations are tedious enough.

THEOREM B. *Let us consider Eq.* (1) *or, equivalently* (2), *where* $F_k^{\pm}$ *is defined like in* (3). *Assume that* $F_1^{+}(z, \bar{z}) = (i + \lambda^{+})z$ (*resp.* $F_1^{-}(z, \bar{z}) = (i + \lambda^{-})z$) *with* $\lambda^{\pm} \in \mathbb{R}$, *when the component equation* $\dot{z} = F^{+}(z, \bar{z})$ (*resp.* $\dot{z} = F^{-}(z, \bar{z})$) *in* (2) *is of type* (F). *Then its first Lyapunov constants at the origin are the following.*

(a) *If* $(0,0)$ *is a singularity of focus–focus type, then*

$$V_1 = e^{\pi(\lambda^{+} + \lambda^{-})} - 1,$$
$$V_2 = e^{-\lambda^{+}\pi} w_2^{+}(\pi) + w_2^{-}(\pi) e^{2\lambda^{+}\pi},$$
$$V_3 = e^{-\lambda^{+}\pi} w_3^{+}(\pi) - 2e^{-2\lambda^{+}\pi}\left(w_2^{+}(\pi)\right)^2 + w_3^{-}(\pi) e^{3\lambda^{+}\pi}.$$

(b) *If* $(0,0)$ *is a singularity of focus–parabolic type, then*

$$V_1 = e^{\lambda^{+}\pi} - 1, \qquad V_2 = w_2^{+}(\pi) - \mu_1^{-}, \qquad V_3 = w_3^{+}(\pi) - (\mu_1^{-})^2.$$

(c) *If* $(0,0)$ *is a singularity of parabolic–parabolic type, then*

$$V_1 = 0, \qquad V_2 = \mu_1^+ - \mu_1^-, \qquad V_3 = 0.$$

where $w_2^\pm(\pi)$, $w_3^\pm(\pi)$, $\mu_1^\pm$, *and* $\mu_2^\pm$ *are given by the expressions*

$$\begin{aligned} w_2^\pm(\pi) &= e^{\lambda^\pm \pi}(-e^{\lambda^\pm \pi} - 1)\operatorname{Re}\left[(1 + \lambda^\pm i)\alpha^\pm\right], \\ w_3^\pm(\pi) &= e^{\lambda^\pm \pi}(e^{2\lambda^\pm \pi} - 1) \\ &\quad \times\left\{\operatorname{Re}\left[(1+\lambda^\pm i)\beta^\pm + \lambda^\pm \gamma^\pm\right] - \operatorname{Im}\left[(1+\lambda^\pm i)\delta^\pm\right]\right\} \\ &\quad + e^{-\lambda^\pm \pi}\left(w_2^\pm(\pi)\right)^2, \end{aligned}$$

with

$$\alpha^\pm = \pm\left[\frac{f_{20}^\pm}{i + \lambda^\pm} + \frac{f_{11}^\pm}{-i + \lambda^\pm} + \frac{f_{02}^\pm}{-3i + \lambda^\pm}\right],$$

$$\beta^\pm = \frac{f_{30}^\pm}{2(i + \lambda^\pm)} + \frac{f_{21}^\pm}{2\lambda^\pm} + \frac{f_{12}^\pm}{2(-i + \lambda^\pm)} + \frac{f_{03}^\pm}{2(-2i + \lambda^\pm)},$$

$$\gamma^\pm = \frac{f_{20}^\pm \overline{f_{20}^\pm} + f_{11}^\pm \overline{f_{11}^\pm} + f_{02}^\pm \overline{f_{02}^\pm}}{4\lambda^\pm} + \frac{f_{11}^\pm \overline{f_{20}^\pm} + f_{02}^\pm \overline{f_{11}^\pm}}{2(-i + \lambda^\pm)} + \frac{f_{02}^\pm \overline{f_{20}^\pm}}{2(-2i + \lambda^\pm)},$$

$$\begin{aligned} \delta^\pm = {} & \frac{\left(f_{20}^\pm\right)^2}{4(i + \lambda^\pm)} + \frac{f_{20}^\pm f_{11}^\pm}{2\lambda^\pm} + \frac{2 f_{20}^\pm f_{02}^\pm + \left(f_{11}^\pm\right)^2}{4(-i + \lambda^\pm)} + \frac{f_{11}^\pm f_{02}^\pm}{2(-2i + \lambda^\pm)} \\ & + \frac{\left(f_{02}^\pm\right)^2}{4(-3i + \lambda^\pm)}, \end{aligned}$$

$$\mu_1^\pm = \frac{2}{3}\,\frac{a^\pm n^\pm - (b^\pm + m^\pm) l^\pm}{a^\pm l^\pm}.$$

Remark. The techniques used to prove Theorem B can also be used to obtain V_k for $k \geq 4$. In fact, to develop Example C, we compute V_4 and V_5, for a particular family of focus–focus type.

When either λ^+ or λ^- is zero or both coincide, for a singularity of focus–focus type, the expression for the Lyapunov constants are shorter than in the general case. As an illustration we give V_1, V_2, and V_3 in next remark, for the case $\lambda^+ = \lambda^-$.

Remark. Assume that in Theorem B we have $\lambda^+ = \lambda^- = \lambda$. Then, if $(0,0)$ is a singularity of focus–focus type, $V_1 = e^{2\pi\lambda} - 1$, $V_2 = 2\,\mathrm{Im}(-f_{20}^+ + f_{11}^+ + \frac{1}{3}f_{02}^+ - (-f_{20}^- + f_{11}^- + \frac{1}{3}f_{02}^-))$, and $V_3 = \pi(\mathrm{Re}(f_{21}^+ + f_{21}^-) - \mathrm{Im}(f_{20}^+ f_{11}^+ + f_{20}^- f_{11}^-))$.

Observe that if Eq. (1) comes from a smooth vector field, that is, $(X^+, Y^+) = (X^-, Y^-) = (X, Y)$, the above result reproduces the well known values of the first three Lyapunov constants for a critical point of focus type, $V_1 = e^{2\pi\lambda} - 1$, $V_2 = 0$, $V_3 = 2\pi(\mathrm{Re}(f_{21}) - \mathrm{Im}(f_{20}f_{11}))$. See [7, 9, 13, 14], for instance.

To conclude this section we give in next examples some simple applications of the above theorem. These examples show how to use the above results to generate limit cycles for systems of type (1) with singular points of pseudo-focus type.

Note that in the smooth case it is known (see [2, p. 243], for instance) that all the even Lyapunov constants associated with a singularity of focus type are zero while that in the non-smooth case it is not true in general. This fact provokes, for instance, that while for quadratic systems the maximum number of limit cycles that bifurcates from the origin is 3 (using V_1, V_3, V_5, and V_7; see [4]), it suffices to use V_1, V_2, V_3, and V_4 in system (7) to generate also three limit cycles. In fact by a more specific study of system (7) we think that six limit cycles can be generated from the origin (using $V_1, V_2, \ldots, V_7$). We remark that although the discontinuous system (7) can have twice the number of limit cycles than the smooth case, the free parameters of (7) and the ones of the quadratic system $\dot{z} = (i + \lambda)z + f_{20}z^2 + f_{11}z\bar{z} + f_{02}\bar{z}^2$ coincide.

EXAMPLE C (focus–focus case). Let Eqs. (1), (2) be given by the expression

$$\dot{z} = \begin{cases} (i + \lambda)z + f_{20}z^2 + f_{11}z\bar{z} + f_{02}\bar{z}^2, & \text{if } \mathrm{Im}(z) \geq 0, \\ iz, & \text{if } \mathrm{Im}(z) \leq 0, \end{cases} \tag{7}$$

where $f_{20} = -\frac{11}{16} + \frac{2}{\pi}y - \frac{15}{32}z + \frac{6-3x}{8}i$, $f_{11} = \frac{11}{12} + \frac{5}{8}z - \frac{4}{\pi}y + i$, $f_{02} = \frac{37}{48} + \frac{2}{\pi}y - \frac{5}{32}z + \frac{-6+3x}{8}i$, for arbitrary real numbers: λ, x, y, and z. Then, by choosing $\lambda < 0$, $x > 0$, $y < 0$, and $z > 0$ small enough, such that $|\lambda| \ll |x| \ll |y| \ll |z|$, Eq. (7) has, at least, four limit cycles bifurcating from the origin.

The fact that $\dot{z} = 1 + iz + i\bar{z}$ and $\dot{z} = iz$ induce the same half return map h^- in $\mathrm{Im}(z) \leq 0$ allows us to state next result.

EXAMPLE D (focus–parabolic case). Let Eqs. (1), (2) be given by

$$\dot{z} = \begin{cases} (i + \lambda) z + f_{20} z^2 + f_{11} z\bar{z} + f_{02} \bar{z}^2, & \text{if } \operatorname{Im}(z) \geq 0, \\ 1 + iz + i\bar{z}, & \text{if } \operatorname{Im}(z) \leq 0, \end{cases}$$

where λ, f_{20}, f_{11} and f_{02} are as in Example C. Then this system has at least four limit cycles bifurcating from the origin.

EXAMPLE E (parabolic–parabolic case). Let Eqs. (1), (2) be given by the expression

$$\dot{z} = \begin{cases} -2 + \dfrac{1}{2}(\varepsilon + i) z + \dfrac{1}{2}(-\varepsilon + i)\bar{z} - \dfrac{i}{4} z^2 + \dfrac{i}{4}\bar{z}^2, & \text{if } \operatorname{Im}(z) \geq 0, \\ 1 + iz + i\bar{z}, & \text{if } \operatorname{Im}(z) \leq 0. \end{cases} \tag{8}$$

Then, by choosing $\varepsilon < 0$ small enough, Eq. (8) has, at least, one limit cycle bifurcating from the origin.

3. PRELIMINARY RESULTS

Remember that the (R, θ, p, q)-generalized polar coordinates are $x = R^p \operatorname{Cs}(\theta)$, $y = R^q \operatorname{Sn}(\theta)$, where p and q will be fixed afterwards and where $\operatorname{Cs}(\theta)$ and $\operatorname{Sn}(\theta)$ are the solution of the Cauchy problem,

$$\dot{\operatorname{Cs}}(\theta) = -\operatorname{Sn}^{2p-1}(\theta), \qquad \dot{\operatorname{Sn}}(\theta) = \operatorname{Cs}^{2q-1}(\theta),$$
$$\operatorname{Cs}(0) = \sqrt[2q]{\frac{1}{p}}, \qquad \operatorname{Sn}(0) = 0;$$

see [6, 8, 15]. By using the (R, θ, p, q)-generalized polar coordinates, each component of Eq. (2) is converted into

$$\frac{dR}{d\theta} = \frac{\operatorname{Re}\Big(\big((z + \bar{z})^{2q-1} + (-1)^p 2^{2(q-p)}(z - \bar{z})^{2p-1}\big)\dot{z}\Big)/(2^{2q-1}R^{2pq-1})}{\operatorname{Im}\big(((p - q)z + (p + q)\bar{z})\dot{z}\big)/(2R^{p+q})}, \tag{9}$$

where this expression is evaluated on $z = R^p \operatorname{Cs}(\theta) + iR^q \operatorname{Sn}(\theta)$. Of course, this change of variables may not make sense for arbitrary values of (R, θ) and p and q. Our first purpose, when we study the focus and the parabolic case, is to find values for p and q for those former change of variables makes sense, i.e., those for which Eq. (9) can be written (in both cases), in $\operatorname{Im}(z) \geq 0$, as an equation of the type

$$\frac{dR}{d\theta} = \sum_{k=1}^{+\infty} T_k(\theta) R^k, \tag{10}$$

defined for (R, θ) in the set $[0, \alpha) \times [0, 2\pi]$, for some real value $\alpha > 0$. Once we get Eq. (2) written as Eq. (10), we will evaluate its solution, $R(\theta; s)$,

$$R(\theta; s) - s = \sum_{k=1}^{+\infty} w_k(\theta) s^k, \qquad \text{with } w_k(0) = 0 \qquad \text{for all } k \geq 1 \tag{11}$$

(i.e., the solution of (10) such that when $\theta = 0$ takes the value s) when $\theta = \pi$. Hence, by defining $h_\theta^+(s) = R(\theta; s)$ as a "return" function between $[0, \alpha) \times \{0\}$ and $[0, \alpha) \times \{\theta\}$, the behaviour of the solution (11) near $R \equiv 0$ is controlled by the values $w_k(\theta)$, $k \geq 1$.

The next two results, proved in [12], are given to simplify as much as possible the expressions that allows us to compute functions $w_k(\theta)$ that appear in (11).

LEMMA 1. (i) *The change of variables* $r = Re^{-\int_0^\theta T_1(\phi)\, d\phi}$ *transforms differential Eq.* (10) *into the differential equation*

$$\frac{dr}{d\theta} = \sum_{k=2}^{+\infty} R_k(\theta) r^k, \tag{12}$$

where $R_k(\theta) = e^{(k-1)\int_0^\theta T_1(\phi)\, d\phi} T_k(\theta)$.

(ii) *Let us consider the solution of Eq.* (12) *written as*

$$r(\theta; \rho) = \rho + \sum_{k=2}^{+\infty} u_k(\theta) \rho^k, \qquad \textit{with } u_k(0) = 0 \qquad \textit{for all } k \geq 2. \tag{13}$$

Then,

$$R(\theta; s) = s + \sum_{k=1}^{+\infty} w_k(\theta) s^k = e^{\int_0^\theta T_1(\phi)\, d\phi} \left[s + \sum_{k=2}^{+\infty} u_k(\theta) s^k \right]$$

is the solution of (10), *satisfying* $R(0; s) = s$; *i.e.*, $w_1(\theta) = e^{\int_0^\theta T_1(\phi)\, d\phi} - 1$ *and* $w_k(\theta) = e^{\int_0^\theta T_1(\phi)\, d\phi} u_k(\theta)$, *for all* $k \geq 2$.

The next result, inspired by [2], allows us to compute the first values of $u_k(\theta)$ in (13). In the sequel we use the notation $\tilde{f} = \tilde{f}(\theta) = \int_0^\theta f(s)\,ds$.

PROPOSITION 2. *Given Eq.* (12), *the functions* $u_i(\theta)$, $i = 2, \dots, 5$ *involved in its solution*, (13), *are*: $u_2(\theta) = \tilde{R}_2$, $u_3(\theta) = (\tilde{R}_2)^2 + \tilde{R}_3$, $u_4(\theta) = (\tilde{R}_2)^3 + 2\tilde{R}_2\tilde{R}_3 + \widetilde{\tilde{R}_2 R_3} + \tilde{R}_4$, *and* $u_5(\theta) = (\tilde{R}_2)^4 + 3(\tilde{R}_2)^2\tilde{R}_3 + \widetilde{\left((\tilde{R}_2)^2 R_3\right)} + 2\tilde{R}_2\widetilde{(\tilde{R}_2 R_3)} + \frac{3}{2}(\tilde{R}_3)^2 + 2\tilde{R}_2\tilde{R}_4 + 2\widetilde{(R_4\tilde{R}_2)} + \tilde{R}_5$.

4. THE FOCUS CASE

Consider Eq. (2). Let us assume that the component equation in $\mathrm{Im}(z) \geq 0$ is of type (F); that is to say, $a^+ = 0$. We will take ordinary (R, θ)-polar coordinates, i.e., $p = q = 1$, in (11). Hence, if we neglect the plus sign, Eq. (9) becomes

$$\frac{dR}{d\theta} = \left.\frac{\mathrm{Re}(\bar{z}\dot{z})/R}{\mathrm{Im}(\bar{z}\dot{z})/R^2}\right|_{z=\mathrm{R}e^{i\theta}} = \frac{\sum_{k=1}^{+\infty}\mathrm{Re}\big(S_k(\theta)\big)R^k}{\sum_{k=1}^{+\infty}\mathrm{Im}\big(S_k(\theta)\big)R^{k-1}}, \tag{14}$$

where $S_k(\theta) = \bar{z}F_k(z, \bar{z})|_{z=e^{i\theta}} = e^{-i\theta}F_k(e^{i\theta}, e^{-i\theta})$. From the expression of $S_1(\theta) = f_{10} + f_{01}e^{-2\theta i}$ and (5) we get that $\mathrm{Im}(S_1(\theta)) > 0$, which implies that Eq. (14) can be written as in (10). In order to evaluate solution (11) of Eq. (14) when $\theta = \pi$, we will prove next proposition.

PROPOSITION 3. *Let the component equation in* $\mathrm{Im}(z) \geq 0$ *of* (2) *be written in polar coordinates as* (14). *Let* $R(\theta; s)$ *be its solution given by* (11). *Then*,

$$R(\pi; s) - s = w_1(\pi)s + O(s^2),$$

where

$$w_1(\pi) = \exp\left\{\frac{\pi\,\mathrm{Re}(f_{10})}{\sqrt{\left(\mathrm{Im}(f_{10})\right)^2 - f_{01}\overline{f_{01}}}}\right\} - 1.$$

Moreover, *if* $F_1(z, \bar{z}) = (i + \lambda)z$ *then*

$$R(\pi, s) - s = w_1(\pi)s + w_2(\pi)s^2 + w_3(\pi)s^3 + O(s^4),$$

where $w_1(\pi) = e^{\lambda\pi} - 1$, $w_2(\pi) = e^{\lambda\pi}(-e^{\lambda\pi} - 1)\mathrm{Re}[(1 + \lambda i)\alpha]$, *and* $w_3(\pi) = e^{\lambda\pi}(e^{2\lambda\pi} - 1)\{\mathrm{Re}[(1 + \lambda i)\beta + \lambda\gamma] - \mathrm{Im}[(1 + \lambda i)\delta]\} + e^{-\lambda\pi}(w_2(\pi))^2$.

Proof. First we compute $w_1(\pi)$. Consider the expression (14). Direct substitution shows that $w_1(\theta)$ satisfies

$$w_1'(\theta) = \frac{\operatorname{Re}(S_1(\theta))}{\operatorname{Im}(S_1(\theta))}(w_1(\theta) + 1), \qquad w_1(0) = 0.$$

Then

$$w_1(\theta) = \exp\int_0^\theta \frac{\operatorname{Re}(f_{10}) + \operatorname{Re}(f_{01})\cos 2\psi + \operatorname{Im}(f_{01})\sin 2\psi}{\operatorname{Im}(f_{10}) - \operatorname{Re}(f_{01})\sin 2\psi + \operatorname{Im}(f_{01})\cos 2\psi}\, d\psi - 1.$$

Standard integration techniques give that

$$\begin{aligned}
&\int_0^\theta \frac{\operatorname{Re}(f_{10}) + \operatorname{Re}(f_{01})\cos 2\psi + \operatorname{Im}(f_{01})\sin 2\psi}{\operatorname{Im}(f_{10}) - \operatorname{Re}(f_{01})\sin 2\psi + \operatorname{Im}(f_{01})\cos 2\psi}\, d\psi \\
&\quad = -\frac{1}{2}\ln\left|\frac{\operatorname{Im}(f_{10}) - \operatorname{Re}(f_{01})\sin 2\theta + \operatorname{Im}(f_{01})\cos 2\theta}{\operatorname{Im}(f_{10}) + \operatorname{Im}(f_{01})}\right| \\
&\qquad + \frac{\operatorname{Re}(f_{10})}{\sqrt{(\operatorname{Im}(f_{10}))^2 - f_{01}\bar{f}_{01}}} \\
&\qquad \times\left[\arctan\left(\frac{\sqrt{(\operatorname{Im}(f_{10}))^2 - f_{01}\overline{f_{01}}}\ \tan\theta}{\operatorname{Im}(f_{10}) + \operatorname{Im}(f_{01}) - \operatorname{Re}(f_{01})\tan\theta}\right) + f(\theta)\right],
\end{aligned}$$

where $f(\theta)$ is equal either to 0, if $0 \le \theta \le \pi/2$, or to π, if $\pi/2 \le \theta \le \pi$. Hence, the expression of $w_1(\pi)$ follows.

Now we continue computing $w_2(\pi)$ and $w_3(\pi)$ but assuming that $f_{10} = \lambda + i$, $f_{01} = 0$. Observe that in this case $w_1(\theta) = e^{\lambda\theta} - 1$. Note also that Eq. (14) can be written as (10) by the expression

$$\frac{dR}{d\theta} = \frac{\lambda R + \sum_{k=2}^{+\infty} R^k \operatorname{Re}(S_k(\theta))}{1 + \sum_{k=2}^{+\infty} R^{k-1}\operatorname{Im}(S_k(\theta))} = \sum_{k=1}^{+\infty} T_k(\theta) R^k,$$

where $T_1 = \lambda$, $T_2 = \operatorname{Re} S_2 - \lambda \operatorname{Im} S_2$, and $T_3 = \operatorname{Re} S_3 - \operatorname{Re} S_2 \operatorname{Im} S_2 - \lambda \operatorname{Im} S_3 + \lambda(\operatorname{Im} S_2)^2$. By using Lemma 1(i) the above equation is converted into

$$\frac{dr}{d\theta} = \sum_{k=2}^{+\infty} R_k(\theta) r^k,$$

where $R_k = e^{(k-1)\lambda\theta} T_k$, $k \ge 2$.

Therefore, from Proposition 2, when we consider its solution in the form (13), we must compute

$$u_2(\pi) = \tilde{R}_2(\pi) = \operatorname{Re}\int_0^\pi S_2(\psi)e^{\lambda\psi}\,d\psi - \lambda\operatorname{Im}\int_0^\pi S_2(\psi)e^{\lambda\psi}\,d\psi,$$

$$u_3(\pi) = \left(u_2(\pi)\right)^2 + \operatorname{Re}\int_0^\pi\left[S_3(\psi) + \frac{\lambda}{2}\left(-S_2^2(\psi) + S_2(\psi)\overline{S_2}(\psi)\right)\right]$$
$$\times e^{2\lambda\psi}\,d\psi - \operatorname{Im}\int_0^\pi\left[\lambda S_3(\psi) + \frac{1}{2}S_2^2(\psi)\right]e^{2\lambda\psi}\,d\psi.$$

Since by Lemma 1(ii), $w_2(\pi) = e^{\lambda\pi}u_2(\pi)$ and $w_3(\pi) = e^{\lambda\pi}u_3(\pi)$, direct computations give the final expression of $w_2(\pi)$ and $w_3(\pi)$ in terms of the coefficients of F_2 and F_3.

As an example we will explain how to calculate $u_2(\pi)$. From the expression of $S_2(\theta) = f_{20}e^{i\theta} + f_{11}e^{-i\theta} + f_{02}e^{-3\theta i}$ and previous expression of $u_2(\pi)$, we have

$$u_2(\pi) = \operatorname{Re}\left(\int_0^\pi S_2(\psi)e^{\lambda\psi}\,d\psi + \lambda i\int_0^\pi S_2(\psi)e^{\lambda\psi}\,d\psi\right)$$
$$= \operatorname{Re}\left((1+\lambda i)\left[\frac{f_{20}e^{(i+\lambda)\psi}}{i+\lambda} + \frac{f_{11}e^{(-i+\lambda)\psi}}{-i+\lambda} + \frac{f_{02}e^{(-3i+\lambda)\psi}}{-3i+\lambda}\right]_0^\pi\right).$$

∎

5. THE PARABOLIC CASE

Consider Eq. (2). Let us assume that the component equation in $\operatorname{Im}(z) \geq 0$ of (2) is of type (P). Hence, $a^+ \neq 0$. In the next lemma we argue on the best selection of (R, θ, p, q)-generalized polar coordinates for this case.

LEMMA 4. *Let the component equation of* (2) *in* $\operatorname{Im}(z) \geq 0$ *be of type* (P). *Then, the R-power series expansion of* (9) *is well defined if and only if* $q = 2p$.

Proof. If $a^+ \neq 0$, then the expression (9) is written as

$$\frac{dR}{d\theta} = \frac{a^+ \operatorname{Cs}^{2q-1}(\theta) R^{q+1} + \operatorname{Im}(f_{10}^+ + f_{01}^+)\operatorname{Sn}^{2p-1}(\theta)\operatorname{Cs}(\theta) R^{2p+1} + \cdots}{-a^+ q \operatorname{Sn}(\theta) R^q + p \operatorname{Im}(f_{10}^+ + f_{01}^+)\operatorname{Cs}^2(\theta) R^{2p} + \cdots},$$

where the dots indicate the rest of the terms in the R-power series expansion. We note that all of these terms have order greater than those that appear. Consequently, p and q can be neither $q < 2p$ nor $q > 2p$, because in these cases the former expression is written as

$$\frac{dR}{d\theta} = \frac{\operatorname{Cs}^{2q-1}(\theta)}{-q \operatorname{Sn}(\theta)} R + O(R^2), \qquad \text{if } q < 2p,$$

$$\frac{dR}{d\theta} = \frac{\operatorname{Sn}^{2p-1}(\theta)}{p \operatorname{Cs}(\theta)} R + O(R^2), \qquad \text{if } q > 2p.$$

In the case $q = 2p$ we have

$$\frac{dR}{d\theta} = \frac{a^+ \operatorname{Cs}^{2q-1}(\theta) + \operatorname{Im}(f_{10}^+ + f_{01}^+)\operatorname{Sn}^{2p-1}(\theta)\operatorname{Cs}(\theta)}{p\big(-2a^+ \operatorname{Sn}(\theta) + \operatorname{Im}(f_{10}^+ + f_{01}^+)\operatorname{Cs}^2(\theta)\big)} R + O(R^2). \tag{15}$$

From the fact that $a^+ < 0$ and $\operatorname{Im}(f_{10}^+ + f_{01}^+) > 0$ we conclude that the two real roots of $-2a^+ \operatorname{Sn}(\theta) + \operatorname{Im}(f_{10}^+ + f_{01}^+)\operatorname{Cs}^2(\theta) = 0$ lie in $\operatorname{Im}(z) < 0$, which finishes the proof. ∎

As a consequence of the previous lemma, from now on and for simplicity, in the parabolic case we will take $(R, \theta, 1, 2)$-generalized polar coordinates. The next lemma sums up all the previous considerations to the study of the parabolic case. Also, in next lemma, by using a change of variables we get an equivalent expression of Eq. (15), written as Eq. (10), with a simplest $T_1(\theta)$.

LEMMA 5. *Let the component equation of* (1) *in* $\{y \geq 0\}$ *be of type* (P). *Write* (1) *in the new variables* $x_1 = x + \frac{m}{l} y$ *and* $y_1 = \frac{-2a}{l} y$. *Then, in*

$(R, \theta, 1, 2)$-generalized polar coordinates and removing the + sign, it is written as

$$
\begin{aligned}
\frac{dR}{d\theta} &= \sum_{k=1}^{+\infty} T_k(\theta) R^k \\
&= \frac{2\,\mathrm{Sn}(\theta)\mathrm{Cs}(\theta) - \mathrm{Cs}^3(\theta)}{2(\mathrm{Cs}^2(\theta) + \mathrm{Sn}(\theta))} R + \frac{an - l(b+m)}{2al} \\
&\quad \times \frac{\mathrm{Cs}^2(\theta)}{(\mathrm{Cs}^2(\theta) + \mathrm{Sn}(\theta))^2} R^2 + T_3(\theta) R^3 + O(R^4),
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
T_3(\theta) = {} & \frac{1}{4a^2 l^2 (\mathrm{Cs}^2(\theta) + \mathrm{Sn}(\theta))^3} \\
& \times \big[\mathrm{Cs}^5(\theta)(-2adl^2 + 2abln - 2a^2n^2 + 2a^2lq) \\
& \quad + \mathrm{Cs}^3(\theta)\mathrm{Sn}(\theta)(2b^2l^2 - 2adl^2 + cl^3 + 3bl^2m + 2l^2m^2 \\
& \qquad - 2abln - 2almn - al^2o + 2a^2lq) \\
& \quad + \mathrm{Cs}(\theta)\mathrm{Sn}^2(\theta)(cl^3 - bl^2m + 2amln - al^2o) \big].
\end{aligned}
$$

Proof. In order to simplify further expressions we apply the change of variables suggested in [10], which does not change the return map on the x-axis $x_1 = x + \frac{m}{l} y$ and $y_1 = \frac{-2a}{l} y$, to the component equation of (1) in $y \geq 0$. Observe that we obtain

$$
\begin{aligned}
\dot{x} &= a + (b+m)x + \frac{bm - cl}{2a} y + \frac{dl + nm}{l} x^2 + \cdots, \\
\dot{y} &= -2ax - \frac{2an}{l} x^2 + \frac{ol - 2nm}{l} xy + \cdots,
\end{aligned} \tag{17}
$$

if we omit the subscripts "1." If, in addition, we use the $(R, \theta, 1, 2)$-generalized polar coordinates in Eq. (17), and we expand the quotient $\frac{dR}{d\theta}$ in a R-power series like (10), we obtain the expression (16). ∎

Next lemma is a technical result which is a key point for the effective computation of the Lyapunov constants in the parabolic case by using the $(R, \theta, 1, 2)$-generalized polar coordinates.

LEMMA 6. *Let* $\mathrm{Sn}(\theta)$ *and* $\mathrm{Cs}(\theta)$ *be the generalized trigonometric functions introduced in Section* 3. *Then*

$$\int \frac{\mathrm{Cs}^m(\theta)\mathrm{Sn}^l(\theta)}{\left(\mathrm{Cs}^2(\theta)+\mathrm{Sn}(\theta)\right)^n}\,d\theta$$

$$= \sum_{i=0}^{l}(-1)^i\binom{l}{i}\begin{cases} \dfrac{\left(\mathrm{Cs}^2(\theta)+\mathrm{Sn}(\theta)\right)^{l-i-n+1}}{(l-i-n+1)\mathrm{Cs}^{2(l-i-n+1)}(\theta)}+k, \\ \quad \text{if } l-i-n+1\neq -1, \\ \ln\left|\dfrac{\mathrm{Cs}^2(\theta)+\mathrm{Sn}(\theta)}{\mathrm{Cs}^2(\theta)}\right|+k, \\ \quad \text{if } l-i-n+1=-1, \end{cases}$$

for all l, m, *and* n *positive integers whenever* $2n = m + 2l + 3$, *and where* k *stands for the arbitrary integration constant.*

Proof. Applying the change of variables $x = \mathrm{Sn}(\theta)/\mathrm{Cs}^2(\theta)$ to the integral, we get the equivalent expression

$$\int \frac{(1+2x^2)^{(2n-m-2l-3)/4}x^l}{(1+x)^l}\,dx = \int (1+x)^{-l}x^l\,dx,$$

because of the hypothesis. The calculation of last integral is immediate. Hence the proof is finished. ∎

In order to evaluate solution (11) of Eq. (16) when $\theta = \pi$, we will prove the next proposition.

PROPOSITION 7. *Let* $R(\theta; s)$ *be the solution* (11) *of Eq.* (1) *written as Eq.* (16). *Then* $w_1(\pi) = 0$, $w_2(\pi) = \frac{2}{3}\frac{an-(b+m)l}{al}$, *and* $w_3(\pi) = \left(\frac{2}{3}\frac{an-(b+m)l}{al}\right)^2$.

Proof. Using the properties of the $(R, \theta, 1, 2)$-generalized polar coordinates and the expression of $T_1(\theta)$ given in Lemma 5, standard integration techniques give

$$\begin{aligned} e^{(k-1)\int_0^\theta T_1(\phi)\,d\phi} &= e^{\{(k-1)/2\int_0^\theta d/d\psi \log(\mathrm{Cs}^2(\psi)+\mathrm{Sn}(\psi))\,d\psi\}} \\ &= \left(\mathrm{Cs}^2(\theta)+\mathrm{Sn}(\theta)\right)^{-(k-1)/2}. \end{aligned}$$

Hence, from Lemma 1(i), Eq. (16) is converted into Eq. (12), where

$$R_2(\theta) = \frac{an - (b+m)l}{2al} \frac{\mathrm{Cs}^2(\theta)}{\left(\mathrm{Cs}^2(\theta) + \mathrm{Sn}(\theta)\right)^{5/2}} \quad \text{and}$$

$$R_3(\theta) = \frac{T_3(\theta)}{\mathrm{Cs}^2(\theta) + \mathrm{Sn}(\theta)}.$$

Now, the idea is to get the solution of Eq. (12) in the form (13), by means of Proposition 2. Then, from Lemma 1(ii), and using the fact that $e^{\int_0^\theta T_1(\phi)\,d\phi}|_{\theta=\pi} = 1$, we will have $w_k(\pi) = u_k(\pi)$ for all $k = 2, 3, \ldots,$ which will finish the proof.

From Proposition 2, we must compute

$$u_2(\theta) = \tilde{R}_2(\theta) = \frac{an - (b+m)l}{2al} \int_0^\theta \frac{\mathrm{Cs}^2(\phi)}{\left(\mathrm{Cs}^2(\phi) + \mathrm{Sn}(\phi)\right)^{5/2}}\, d\phi.$$

From Lemma 6, we have that

$$u_2(\theta) = \frac{an - (b+m)l}{3al} \left(1 - \frac{\mathrm{Cs}^3(\theta)}{\mathrm{Cs}^2(\theta) + \mathrm{Sn}(\theta)^{3/2}}\right).$$

Therefore,

$$u_2(\pi) = \frac{2}{3} \frac{an - (b+m)l}{al}.$$

To compute $u_3(\pi)$ we need to calculate $\tilde{R}_3(\pi)$, but $R_3(\theta)$ is an even expression on $\mathrm{Sn}(\theta)$ and odd on $\mathrm{Cs}(\theta)$. Hence, $\tilde{R}_3(\pi) = 0$. This gives, from Proposition 2, that $u_3(\pi) = (\tilde{R}_2(\pi))^2$, which finishes the proof. ∎

Remark 8. About the contact order of the solution, $y = y(x)$, of Eq. (1) which passes through $(0,0)$ and $L = \{(x, y),\ y = 0\}$, for a component equation of type (P) in $y \geq 0$, it is not difficult to see that $(d^2y/dx^2)|_{(0,0)} = l/a^2$ and, consequently, that whenever $l \neq 0$, this solution has a second order contact with L. If $l = 0$ then, taking $n = 0$ and $q > 0$, the flow keeps turning around the origin $(0,0)$. In this case, the contact order becomes four because $(d^3y/dx^3)|_{(0,0)} = 0$ and $(d^4y/dx^4)|_{(0,0)} = 6q/a^2$. In this situation it is not possible to get Eq. (11) by using the change to $(R, \theta, p, 2p)$-generalized polar coordinates suggested in Lemma 4, but it is not difficult to prove that the suitable change to (R, θ, p, q)-generalized polar coordinates is obtained by taking $q = 4p$.

6. PROOF OF THE MAIN RESULTS

Proof of Proposition A and Theorem B. Let us consider Eq. (1), or equivalently (2), and let us apply to each component equation the suitable change of (R, θ, p, q)-polar coordinates, according Section 4 and Lemma 4. Following the ideas given in Section 3, to prove our results we will compose the map induced by the flow of $\dot{z} = F^{+}(z, \bar{z})$ between $\theta = 0$ and $\theta = \pi, h^{+}$, and the map induced by the flow of $\dot{z} = F^{-}(z, \bar{z})$ between $\theta = \pi$ and $\theta = 2\pi, h^{-}$. Their expansions are given by expression (11), with $h^{\pm}(s) = R^{\pm}(\pi, s)$, and are written as

$$\begin{aligned} h^{\pm}(s) &= \left(w_1^{\pm}(\pi) + 1\right)s + w_2^{\pm}(\pi)s^2 + w_3^{\pm}(\pi)s^3 + O(s^4) \\ &= h_1^{\pm}s + h_2^{\pm}s^2 + h_3^{\pm}s^3 + O(s^4). \end{aligned}$$

We note that, in this last expression, the map h^{-} coincides with the return map induced by the flow of $\dot{z} = -F^{-}(-z, -\bar{z})$ between $\theta = 0$ and $\theta = \pi$. Hence the stability of the origin is controlled by the sign of $h^{-}(h^{+}(s)) - s$, where

$$\begin{aligned} h^{-}(h^{+}(s)) - s = (h_1^{-}h_1^{+} - 1)s &+ \left(h_1^{-}h_2^{+} + h_2^{-}(h_1^{+})^2\right)s^2 \\ &+ \left(h_1^{-}h_3^{+} + 2h_2^{-}h_1^{+}h_2^{+} + h_3^{-}(h_1^{+})^3\right)s^3 + O(s^4) \end{aligned}$$

and the three first Lyapunov constants of (2) are

$$V_1 = h_1^{-}h_1^{+} - 1, \qquad V_2 = \frac{h_2^{+} + h_2^{-}(h_1^{+})^3}{h_1^{+}},$$

$$V_3 = \frac{h_1^{+}h_3^{+} - 2(h_2^{+})^2 + h_3^{-}(h_1^{+})^5}{(h_1^{+})^2}.$$

Let us argue about the sign of h_1^{+}. In the focus case, using Proposition 3, $h_1 = e^{\lambda\pi} > 0$, because $w_1(\pi) = e^{\lambda\pi} - 1$. In the parabolic case, using Proposition 7, $h_1 = 1 > 0$, because $w_1(\pi) = 0$. Hence, we obtain the values V_i, $i = 1, 2$, and 3, depending on the singularity type of the origin. Whence, the proof follows. ∎

Proof of Example C. Lyapunov constants V_1, V_2, and V_3, of the origin of (7), follow from Theorem B(a) by taking into account, among other facts, that $\lambda^{-} = 0$ and $w_i^{-}(\theta) \equiv 0$ for all $i \geq 2$, which simplifies their calculation. In fact, we have $V_i = \omega_i^{+}(\pi)$, for all $i \geq 2$. Hence, $V_1 = e^{\lambda\pi} - 1$, $V_2 = x$, and $V_3 = y$. In order to generate four limit cycles we need to know the fourth and fifth Lyapunov constants, V_4 and V_5, of the origin of Eq. (7). Their calculation is only interesting if $V_1 = V_2 = V_3 = 0$. Hence, we assume $\lambda = x = y = 0$. Let us calculate V_4 and V_5. It is not difficult, following Lemma 5, to get $T_4 = \mathrm{Re}(S_2)(\mathrm{Im}(S_2))^2$ and $T_5 = -\mathrm{Re}(S_2)(\mathrm{Im}(S_2))^3$. Now, using the fact that $\lambda = 0$, we have $R_k = T_k$ if $k \geq 2$. In addition, from Proposition 2, we get the expression of $u_4^+(\theta)$ and $u_5^+(\theta)$. Hence, from Proposition 1(ii), using again that $\lambda = 0$, we have that $\omega_k^+(\theta) = u_k^+(\theta)$ if $k \geq 2$. Following the same ideas exposed in the proof of Theorem B we have $V_4 = \omega_4^+(\pi) = z$ and, in the case $V_4 = 0$ (i.e., if $z = 0$), $V_5 = \omega_5^+(\pi) = -\frac{55}{384}\pi$.

Hence, the return map associated to the origin of Eq. (7) is written as

$$\begin{aligned} h(s;\lambda,x,y,z) &\\ &= (e^{\lambda\pi}-1)s + \big(x + f_2(\lambda,x,y,z)\big)s^2 + \big(y + f_3(\lambda,x,y,z)\big)s^3 \\ &\quad + \big(z + f_4(\lambda,x,y,z)\big)s^4 + \left(-\frac{55}{384}\pi + f_5(\lambda,x,y,z)\right)s^5 \\ &\quad + O(s^6), \end{aligned}$$

for small enough values of the variable s, $s > 0$, where f_i, $i = 2,\dots,5$, are analytic real functions satisfying $f_2(0,x,y,z) \equiv 0$, $f_3(0,0,y,z) \equiv 0$, $f_4(0,0,0,z) \equiv 0$, and $f_5(0,0,0,0) = 0$. By choosing λ, x, y, and z small enough and satisfying the conditions of this example, it is easy to find four values of s for which h vanishes. More concretely, if we consider $h(s;0,0,0,0)$, we can get a value of s, s_0, small enough, for which $h(s_0;0,0,0,0) < 0$. Because of the continuity of the function h, it is possible to assure that $h(s_0;0,0,0,z) < 0$, for small enough values of z. Let $z_1 > 0$ be one such value of z. In this situation, we have that $h(s;0,0,0,z_1) = z_1 s^4 + O(s^5)$. Hence, we can have a value of s, s_1, small enough and smaller than s_0, for which $h(s_1;0,0,0,z_1) > 0$. Whence, between s_1 and s_0, we obtain the first change of sign of h. Arguing in a similar way, we can get the four changes of sign of h. See also [5]. ∎

Proof of Example E. Lyapunov constants V_1, V_2, and V_3, of the origin of Eq. (8) follow straightaway from Theorem B(c). V_4 needs some more computations. They are $V_1 = 0$, $V_2 = \frac{\varepsilon}{3}$, $V_3 = 0$, and $V_4 = \frac{1}{30}$. Hence, for the choice of the parameter ε made in the statement of this lemma and arguing as in the proof of Example C, we can create one unstable small limit cycle around the origin as we wanted to prove. ∎

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